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(6) SYSTEMS OF GENERALIZED ABEL INTEGRAL EQUATIONS WITH APPLICATIONS TO SIMULTANEOUS DUAL RELATIONS.

by

(9) Interim rept.,

(10) J. R. Walton[1)]

Department of Mathematics
Texas A&M University
College Station, Tx.

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ABSTRACT

A method is presented for solving certain systems of generalized Abel integral equations by constructing equivalent singular integral equations and their corresponding Riemann boundary value problems. An application is then given to a class of simultaneous dual relations of a type arising in bimedia fracture problems in elasticity. The equations discussed in this paper generalize those considered in an earlier paper of Lowengrub and Walton [3].

1. Introduction

In this paper we describe a method for solving systems of generalized Abel integral equations of the type

$$(1)\quad \begin{aligned} a_1(x^p) \int_0^x \frac{\alpha_1(t^p)\phi_1(t)}{(x^p-t^p)^{\mu_1}}\,dt + b_2(x^p) \int_x^1 \frac{\beta_2(t^p)\phi_2(t)}{(t^p-x^p)^{\mu_2}}\,dt &= f_1(x) \\ & \qquad 0 < x < 1 \\ b_1(x^p) \int_x^1 \frac{\beta_1(t^p)\phi_1(t)}{(t^p-x^p)^{\mu_1}}\,dt + a_2(x^p) \int_0^x \frac{\alpha_2(t^p)\phi_2(t)}{(x^p-t^p)^{\mu_2}}\,dt &= f_2(x)\ . \end{aligned}$$

Since only the cases $p = 1$ or $p = 2$ occur in applications, we shall restrict the subsequent discussion to those cases.

The equations (1) are a generalization of those analyzed in [3] for which $\mu_1 = \mu_2$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$. That reference also includes a discussion of an application of such systems to problems in elasticity. In particular, a method was presented in [3] for reducing a simultaneious set of dual relations involving Hankel transforms to a simultaneous system of fractional integral and differential operators. Under certain conditions the systems obtained in that way were shown to be equivalent to one of the systems of Abel type equations for which closed form solutions were constructed in [3]. However, the conditions that must be imposed upon such simultaneous dual relations to yield Abel systems within the scope of the techniques of [3] are very restrictive.

In contrast, the method presented here is applicable to a very large class of simultaneous dual relations. Unfortunately this is at the expense of obtaining closed form solutions. What is achieved is a transformation of the dual relations to a system of singular integral equations with Cauchy dominant singular part. Such systems have been studied extensively. (See [2], [4].) For example, the Noether theorems [4] answer the questions of existence and uniqueness of solutions to such systems, and these systems are known to be equivalent to certain Fredholm integral equations of the second kind. Recently, equations (1) with $p = 1$ have arisen in the study of certain bimedia fracture problems for power law viscoelastic solids. This application will be the subject of a future paper.

Very few theoretical results for simultaneous dual relations have appeared in the literature. The method described in this paper provides a means of pursuing such investigations and is useful for obtaining insight into the nature of simultaneous dual relations. For example, applying the Noether theorems to the associated singular integral equations, yields conditions necessary for the existence of a unique solution. We do not attempt to present a rigorous analysis of the dual relations considered here. However, it should be straight forward, albeit tedious, to do so by justifying our formal manipulations within the distributional framework employed in [6] and [7] or that developed by Braaksma and Schuitman [1].

In Sections 2 and 3 we consider (1) for $p = 1$ and $p = 2$ respecitvely. Section 4 contains the application to simultaneous dual relations.

2. First Abel System

In this section we consider the generalized Abel system (1) with $p = 1$. It is assumed that

$$\text{(2)} \qquad \alpha_i(t) = \alpha_i{}^*(t)t^{\nu_i}$$

$$\text{and} \quad \beta_i(t) = \beta_i{}^*(t)(1-t)^{\lambda_i} \qquad i = 1,2$$

where $\alpha_i{}^*(t)$ and $\beta_i{}^*(t)$ are continuously differentiable on $[0,1]$ and non-vanishing at the endpoints. It will prove convenient to introduce the following notation:

$$\text{(3)} \qquad \nu_i{}' = \min(0,\nu_i) \ , \ \lambda_i{}' = \min(0,\lambda_i),$$

$$\text{(4)} \qquad \begin{aligned} \tilde{\alpha}_i(t) &= \alpha_i(t)t^{-\nu_i{}'}(1-t)^{-\lambda_i} \ , \\ \tilde{\beta}_i(t) &= \beta_i(t)t^{-\nu_i{}'}(1-t)^{-\lambda_i{}'} \ , \end{aligned}$$

$$\text{(5)} \qquad \tilde{\phi}_i(t) = \phi_i(t)t^{\nu_i{}'}(1-t)^{\lambda_i{}'} \ ,$$

$$\text{(6)} \qquad \begin{aligned} I_i(\phi_i) &= \int_0^x \frac{\alpha_i(t)\phi_i(t)}{(x-t)^{\mu_i}}\,dt \ , \\ K_i(\phi_i) &= \int_x^1 \frac{\beta_i(t)\phi_i(t)}{(t-x)^{\mu_i}}\,dt \ , \end{aligned}$$

$$\text{(7)} \qquad h_i(x) = \int_0^x \frac{\phi_i(t)}{(x-t)^{\mu_i}}\,dt \ ,$$

$$\text{(8)} \qquad k_i(x) = \int_x^1 \frac{\phi_i(t)}{(t-x)^{\mu_i}}\,dt \ ,$$

(9) $$R_i(z) = [z(1-z)]^{\frac{1-\mu_i}{2}}$$

and

(10) $$\phi_i(z) = [R_i(z)]^{-1} \int_0^1 \frac{\tilde{\phi}_i(t)}{(t-z)^{\mu_i}}\, dt \quad .$$

We seek solutions $\phi_i(t)$, $i = 1,2$ such that

(11) $$\tilde{\phi}_i(t) = \frac{\phi_i{*}(t)}{[t(1-t)]^{1-\mu_i-\varepsilon}}$$

where $\phi_i{*}t)$ is Hölder continuous on $[0,1]$ and ε is a positive number. It follows that when a suitable branch is chosen for the multivalued function $[R_i(z)(t-z)^{\mu_i}]^{-1}$, $\phi_i(z)$ is analytic in the complex plane cut along $[0,1]$ and satisfies the asymptotic estimates

$$\phi_i(z) = 0(\tfrac{1}{z}) \quad \text{as} \quad z \to \infty$$

$$\phi_i(z) = 0(z^{\frac{\mu_i-1}{2}}) \quad \text{as} \quad z \to 0$$

(12) and

$$\phi_i(z) = 0((1-z)^{\frac{\mu_i-1}{2}}) \quad \text{as} \quad z \to 1 \quad .$$

We remark also that $h_i(0+) = k_i(1-) = I_i(\phi_i)(0+) = K_i(\phi_i)(1-) = 0$ and $h_i(1-)$, $k_i(0+)$, $I_i(\phi_i)(1-)$ and $K_i(\phi_i)(0+)$ are all finite.

Let $\Phi_i^{\pm}(x)$ denote the following limits:

$$\Phi_i^{+}(x) = \lim_{\substack{z \to x \\ I_m(z)>0}} \Phi_i(z)$$

$$0<x<1$$

$$\Phi_i^{-}(x) = \lim_{\substack{z \to x \\ I_m(z)<0}} \Phi_i(z) \quad .$$

It is then readily verified that

(13) $$h_i(x) = \left[\frac{e^{\mu_i \Pi i}\Phi_i^{+}(x)+\Phi^{-}(x)}{(e^{2\mu_i \Pi i}-1)}\right] R_i(x) \quad \text{and}$$

(14) $$k_i(x) = -\left[\frac{\Phi_i^{+}(x)+e^{\mu_i \Pi i}\Phi_i^{-}(x)}{(e^{2\mu_i \Pi i}-1)}\right] R_i(x) \quad .$$

From lines (2) - (5) we observe that

(15) $$\tilde{\phi}_i(t) = \frac{\sin \mu_i \Pi}{\Pi} \int_0^t \frac{h_i'(y)dy}{(t-y)^{1-\mu_i}}$$

(16) $$= -\frac{\sin \mu_i \Pi}{\Pi} \int_t^1 \frac{k_i'(y)dy}{(y-t)^{1-\mu_i}} \quad .$$

Moreover, from (15) we obtain

$$I_i(\phi_i)(x) = \int_0^x \frac{\alpha_i(t)\phi_i(t)dt}{(x-t)^{\mu_i}}$$

$$= \int_0^x \frac{\tilde{\alpha}_i(t)\tilde{\phi}_i(t)dt}{(x-t)^{\mu_i}}$$

$$= \frac{\sin \mu_i \Pi}{\Pi} \int_0^x \frac{\tilde{\alpha}_i(t)dt}{(x-t)^{\mu_i}} \int_0^t \frac{h_i'(y)dy}{(t-y)^{1-\mu_i}}$$

$$= \frac{\sin \mu_i \Pi}{\Pi} \int_0^x h_i'(y)dy \int_y^x \frac{\tilde{\alpha}_i(t)dt}{(x-t)^{\mu_i}(t-y)^{1-\mu_i}}$$

(17) $$= h_i(x)\tilde{\alpha}_i(x) - \frac{\sin \mu_i \Pi}{\Pi} \int_0^x h_i(y)K_{1,i}(x,y)dy$$

where

$$K_{1,i}(x,y) = \int_0^1 \frac{\tilde{\alpha}_i'(s(x-y)+y)ds}{(1-s)^{\mu_i-1}s^{1-\mu_i}} \quad .$$

It should be observed that $K_{1,i}(x,y)$ is continuous for $0 < y < x < 1$ and integrable on the triangle $0 \le y \le x \le 1$.

Similarly, as a consequence of (16), we have

(18) $$K_i(\phi_i)(x) = k_i(x)\tilde{\beta}_i(x) + \frac{\sin \mu_i \Pi}{\Pi} \int_x^1 k_i(y) K_{2,i}(x,y)dy$$

where

$$K_{2,i}(x,y) = \int_0^1 \frac{\tilde{\beta}_i'(s(y-x)+y)ds}{(1-s)^{\mu_i-1}s^{1-\mu_i}} \quad ,$$

and $K_{2,i}(x,y)$ is continuous for $0 < x < y < 1$ and integrable on $0 \le x \le y \le 1$.

Substitution of (13), (14), (17), and (18) into (1) yields the system

$$f_i(x) = \frac{a_i(x)\tilde{\alpha}_1(x)R_1(x)}{(e^{2\mu_1\Pi_i}-1)}\left[e^{\mu_1\Pi i}\phi_1^{+}(x) + \phi_1^{-}(x)\right]$$

$$- \frac{b_2(x)\tilde{\beta}_2(x)R_2(x)}{(e^{2\mu_2\Pi i}-1)}\left[\Phi_2^{+}(x) + e^{\mu_2\Pi i}\phi_2^{-}(x)\right]$$

(19)

$$- \frac{a_i(x)\sin\mu_1\Pi}{(e^{2\mu_1\Pi i}-1)\Pi}\int_o^x\left[e^{\mu_1\Pi i}\phi_1^{+}(y) + \phi_1^{-}(y)\right]R_1(y)K_{1,1}(x,y)dy$$

$$- \frac{b_2(x)\sin\mu_2\Pi}{(e^{2\mu_2\Pi i}-1)\Pi}\int_x^1\left[\Phi_2^{+}(x)+e^{\mu_2\Pi i}\phi_2^{-}(x)\right]R_2(y)K_{2,2}(x,y)dy$$

$$f_2(x) = -\frac{b_1(x)\tilde{\beta}_i(x)R_1(x)}{(e^{2\mu_2\Pi i}-1)}\left[\phi_1^{+}(x) + e^{\mu_1\Pi i}\phi_1^{-}(x)\right]$$

(20)

$$+ \frac{a_2(x)\tilde{\alpha}_2(x)R_2(x)}{(e^{2\mu_2\Pi i}-1)}\left[e^{\mu_2\Pi i}\phi_2^{+}(x) + \phi_2^{-}(x)\right]$$

$$- \frac{b_1(x)\sin\mu_1\Pi}{(e^{2\mu_1\Pi i}-1)\Pi}\int_x^1\left[\phi_1^{+}(y) + e^{\mu_1\Pi i}\phi_1^{-}(y)\right]R_1(y)K_{2,1}(x,y)dy$$

$$- \frac{a_2(x)\sin\mu_2\Pi}{(e^{2\mu_2\Pi i}-1)\Pi}\int_o^x\left[e^{\mu_2\Pi i}\phi_2^{+}(y) + \phi_2(y)\right]R_2(y)K_{1,2}(x,y)dy\ .$$

It is straight forward to verify that solving the Abel system (1) in the class (11) is equivalent to solving the generalized Riemann system (19) and (20), i.e. to finding sectionally analytic functions $\Phi_i(z)$, $i = 1,2$, satisfying (12) and the generalized boundary equations (19) and (20).

The generalized Riemann problem (19) and (20) may be transformed, in turn, into an equivalent system of singular integral equations with Cauchy dominant singular part by a method outlined in Gakhov [2]. In particular, define

$$(21) \qquad \Psi_i(z) = \frac{1}{2\pi i} \int_0^1 \frac{\psi_i(t)dt}{(t-z)} ,$$

and recall the Plemelj formulas [4]

$$(22) \qquad \Psi_i^{\pm}(x) = \pm \frac{1}{2} \psi_i(x) + \frac{1}{2\pi i} \int_0^1 \frac{\psi_i(t)dt}{(t-x)} .$$

Substitution of (21) into (19) and (20) yields a system of singular integral equations, which in matrix notation becomes

$$(23) \qquad K^\circ(\psi) + k(\psi) = f ,$$

where $\psi = (\psi_1,\psi_2)^T$ and $f = (f_1,f_2)^T$. The dominant singular part of (23), $K^\circ(\psi)$, is given by

$$K^\circ(\psi) = A(x)\psi(x) + B(x) \frac{1}{\pi i} \int_0^1 \frac{\psi(t)dt}{(t-x)} ,$$

where $A(x) = (a_{ij}(x))$, $B(x) = (b_{ij}(x))$ and

$$a_{11}(x) = \frac{a_1(x)\tilde{\alpha}_1(x)R_1(x)}{2(e^{\mu_1 \pi i}+1)}$$

$$a_{12}(x) = \frac{b_2(x)\tilde{\beta}_2(x)R_2(x)}{2(e^{\mu_2 \pi i}+1)}$$

$$a_{21}(x) = \frac{b_1(x)\tilde{\beta}_1(x)R_1(x)}{2(e^{\mu_1 \pi i}+1)}$$

$$a_{22}(x) = \frac{a_2(x)\tilde{\alpha}_2(x)R_2(x)}{2(e^{\mu_2 \pi i}+1)}$$

$$b_{11}(x) = \frac{a_1(x)\tilde{\alpha}_1(x)R_1(x)}{2(e^{{}_1\mu i\pi}-1)}$$

$$b_{12}(x) = \frac{-b_2(x)\tilde{\beta}_2(x)R_2(x)}{2(e^{\mu_2 \pi i}-1)}$$

$$b_{21}(x) = \frac{-b_1(x)\tilde{\beta}_1(x)R_1(x)}{2(e^{\mu_1 \pi i}-1)}$$

$$b_{22}(x) = \frac{a_2(x)\tilde{\alpha}_2(x)R_2(x)}{2(e^{\mu_2 \pi i}-1)} \quad .$$

The operator $k(\psi)$ in (23) is easily seen to be compact. The theory of systems of the type (23) is well known. (See, for example, Muskhelishvili [4].) In particular it is known that (23) is equivalent to a system of Fredholm equations of the second kind. In this sense we may regard the above analysis as providing a solution of (1), although except for special cases, the solution is not obtainable in closed form. We remark further, that when $\mu_1 = \mu_2$, the dominant singular part of (23), $K^\circ(\psi)$, may be substantially simplified. Important theoretical information about (1) may be obtained from (23). However, we shall postpone a consideration of this until Section 4 when an application of (1) to dual relations is discussed.

Section 4. Second Abel System.

We next consider the system (1) with $p = 2$. The technique employed for this case is in the same spirit as that of the previous section, with only slight modifications made necessary by the substitution of the non-univalent function z^2 for z.

As in Section 2, it is convenient to introduce certain notation. Assumptions (2) and definitions (3) and (4) are unchanged. Whereas, lines (5) - (10) are replaced by:

$$\tilde{\phi}_i(t) = \phi_i(t)^{2\nu_i'}(1-t^2)^{\lambda_i'} , \tag{24}$$

$$\begin{aligned} I_i(\phi_i) &= \int_0^x \frac{\alpha_i(t^2)\phi_i(t)dt}{(x^2-t^2)^{\mu_i}} \\ K_i(\phi_i) &= \int_x^1 \frac{\beta_i(t^2)\phi_i(t)dt}{(t^2-x^2)^{\mu_i}} , \end{aligned} \tag{25}$$

$$h_i(x) = \int_0^x \frac{\tilde{\phi}_i(t)dt}{(x^2-t^2)^{\mu_i}} , \tag{26}$$

$$(27) \qquad k_i(x) = \int_x^1 \frac{\phi_i(t)dt}{(t^2-x^2)^{\mu_i}} \quad ,$$

$$(28) \qquad R_i(z) = [z^2-1]^{\frac{1}{2}-\mu_i}$$

and

$$(29) \qquad \Phi_i(z) = \frac{1}{R_i(z)} \int_0^1 \frac{\tilde{\phi}_i(t)dt}{(z^2-t^2)^{\mu_i}} \quad .$$

Instead of (11), we now seek solutions $\phi_i(t)$ such that

$$(30) \qquad \tilde{\phi}_i(t)t^{-\mu_i} = \frac{\phi_i^*(t)}{[t(1-t)]^{1-\mu_i-\varepsilon}} \quad .$$

When a suitable branch is chosen for $(z^2-t^2)^{-\mu_i}/R_i(z)$, we see from (30), that $\Phi_i(z)$ is analytic in the complex plane cut along $[-1,1]$ and satisfies

$$\Phi_i(z) = O(\tfrac{1}{z}) \quad \text{as} \quad z \to \infty$$

(31)

and $\quad \Phi_i(z) = O((z^2-1)^{\mu_i-\frac{1}{2}}) \quad \text{as} \quad z \to \pm 1$.

Moreover, we may conclude that $h_i(0+) = k_i(1-) = I_i(\phi_i)(0+) = K_i(\phi_i)(1-) = 0$ and $h_i(1-)$, $k_i(0+)$, $I_i(\phi_i)(1-)$ and $K_i(\phi_i)(0+)$ are all finite.

The limits $\Phi^{\pm}(x)$ are defined as before, only now they are computed for $-1 < x < 1$. It is easy to show that when $0 < x < 1$,

$$(32) \qquad h_i(x) = [\Phi_i^+(x) + \Phi_i^-(x)] \frac{(1-x^2)^{\frac{1}{2}-\mu_i}}{2 \sin \mu_i \Pi}$$

and

$$(33) \qquad k_i(x) = -[e^{-\mu_i \Pi i}\Phi_i^+(x) + e^{\mu_i \Pi i}\Phi_i^-(x)] \frac{(1-x^2)^{\frac{1}{2}-\mu_i}}{2 \sin \mu_i} ,$$

and when $-1 < x < 0$

$$(34) \qquad \Phi_i^{\pm}(x) = - \overline{\Phi^{\pm}(-x)} .$$

From (26) and (27) we obtain

$$(35) \qquad \tilde{\phi}_i(t) = \frac{\sin \mu_i \Pi}{\Pi} \, 2t \int_0^t \frac{h_i'(y)dy}{(t^2-y^2)^{1-\mu_i}}$$

$$(36) \qquad = - \frac{\sin \mu_i}{\Pi} \, 2t \int_t^1 \frac{k_i'(y)dy}{(y^2-t^2)^{1-\mu_i}} .$$

Corresponding to (17) and (18), substitution of (35) and (36) into (25) yields

$$(37) \qquad I_i(\phi_i) = h_i(x)\tilde{\alpha}_i(x^2) - \frac{\sin \mu_i \Pi}{\Pi} \int_0^x h_i(y)yK_{1,i}(x,y)dy$$

and

$$(38) \qquad K_i(\phi_i) = k_i(x)\tilde{\beta}_i(x^2) + \frac{\sin \mu_i \Pi}{\Pi} \int_x^1 k_i(y)yK_{2,i}(x,y)dy$$

where

$$(39) \qquad K_{1,i}(x,y) = 2\int_0^1 \frac{\tilde{\alpha}_i'(\sigma(x^2-y^2)+y^2)d\sigma}{(1-\sigma)^{\mu_i-1}\sigma^{1-\mu_i}}$$

and

$$(40) \qquad K_{2,i}(x,y) = 2\int_0^1 \frac{\tilde{\beta}_i'(\sigma(x^2-y^2)+y^2)d\sigma}{(1-\sigma)^{\mu_i-1}\sigma^{1-\mu_i}} .$$

Substitution of (37) and (38) into (1) yields a Riemann boundary system valid for $0 < x < 1$ which in matrix notation becomes

$$(41) \qquad A(x)\,\Phi^+(x) + B(x)\,\Phi^-(x) + K(\Phi^+) + H(\Phi^-) = f(x)$$

where

$$\Phi(z) = (\Phi_1(z)\ ,\ \Phi_2(z))^T ,$$

$$f(x) = (f_1(x)\ ,\ f_2(x))^T ,$$

$$A(x) = (a_{ij}(x))\ ,\ B(x) = \overline{A(x)}\ ,\quad K = (k_{ij}) \quad \text{and} \quad H = (h_{ij})$$

with

$$a_{11}(x) = \frac{a_1(x^2)\tilde{\alpha}_1(x^2)}{2\sin\mu_1\Pi}(1-x^2)^{\frac{1}{2}-\mu_1} ,$$

$$a_{12}(x) = -\frac{b_2(x^2)\tilde{\beta}_2(x^2)}{2\sin\mu_2\Pi}(1-x^2)^{\frac{1}{2}-\mu_2}e^{-\mu_2\Pi i} ,$$

$$a_{21}(x) = -\frac{b_1(x^2)\tilde{\beta}_1(x^2)}{2\sin\mu_1\Pi}(1-x^2)^{\frac{1}{2}-\mu_1}e^{-\mu_1\Pi i} ,$$

$$a_{22}(x) = \frac{a_2(x^2)\tilde{\alpha}_2(x^2)}{2 \sin \mu_2 \Pi} (1-x^2)^{\frac{1}{2} - \mu_2} ,$$

$$k_{11}(\phi) = - \frac{a_1(x^2)}{2\Pi} \int_0^x \phi(y)(1-y^2)^{\frac{1}{2} - \mu_1} yK_{11}(x,y)dy ,$$

$$k_{12}(\phi) = - \frac{b_2(x^2)}{2\Pi} - e^{-\mu_2 \ell i} \int_x^1 \phi(y)(1-y^2)^{\frac{1}{2} - \mu_2} yk_{2,2}(x,y)dy ,$$

$$k_{21}(\phi) = - \frac{b_1(x^2)}{2\Pi} e^{-\mu_1 \Pi i} \int_x^1 \phi(y)(1-y^2)^{\frac{1}{2} - \mu_1} yK_{2,1}(x,y)dy$$

and

$$k_{22}(\phi) = - \frac{a_2(x^2)}{2\Pi} \int_0^x \phi(y)(1-y^2)^{\frac{1}{2} - \mu_2} yK_{1,2}(x,y)dy .$$

The kernel of the operator h_{ij} is conjugate to that of k_{ij} .

To establish the equivalence of (1) to a system of Riemann boundary value problems it is necessary to extend the boundary equation (41) to all of $(-1,1)$. However, from (34) and the fact that for $0 < x < 1$ $K(\phi^+) + H(\phi^-)$ is real it is clear how the extension should be effected. Specifically, if we define for $-1 < x < 0$

$$a_i(x) = a_i(-x) ,$$

$$b_i(x) = b_i(-x)$$

$$k_{ij}(\phi)(x) = k_{ij}(\phi)(-x) ,$$

$$\tilde{\alpha}_i(x) = - \tilde{\alpha}_i(-x) ,$$

$$\beta_i(x) = -\ \beta_i(-x)$$

and

$$f_i(x) = f_i(-x)\ ,$$

then

$$A(x) = -\ \overline{A(-x)}$$

and

$$B(x) = -\ \overline{B(-x)}$$

and (41) is valid for $-1 < x < 1$.

By an argument entirely analogous to that of Section 2, the introduction of $\psi_i(t)$ through

$$\Phi_i(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\psi_i(t)dt}{(t-z)}$$

transforms (1) to an equivalent system of singular integral equations with Cauchy dominant singular part. In the next section we consider an application of (1) with $p = 2$ to certain simultaneous dual relations.

4. Simultaneous Dual Relations

In this section we consider an application of the analysis presented in Section 3 to simultaneous dual integral equations of the form

$$\int_0^\infty [a_1 A(\xi) + b_1 B(\xi)]\xi^{-2\alpha} J_\mu(\xi x)d\xi = F_1(x)$$

$$0 < x < 1$$

$$\int_0^\infty [a_2 A(\xi) + b_2 B(\xi)]\xi^{-2\beta} J_\nu(\xi x)d\xi = F_2(x)$$

(42)

$$\int_0^\infty A(\xi) J_\mu(\xi x)d\xi = 0$$

$$1 < x < \infty$$

$$\int_0^\infty B(\xi) J_\nu(\xi x)d\xi = 0 \ ,$$

where a, b, c and d are constants. As was remarked in [3], such systems arise in bimedia fracture problems in elasticity. It was demonstrated in [3] that the system (42) may be transformed into the system

$$a_1 K_{\frac{\mu}{2}-\alpha,\nu-\lambda}[\eta_1] + b_1 I_{\frac{\nu}{2},\frac{\mu-\nu}{2}}[\eta_2] = f_1(x)$$

$$0 < x < 1$$

(43)

$$a_2 I_{\frac{\mu}{2},\frac{\nu-\mu}{2}}[\eta_1] + b_2 K_{\frac{\nu}{2}-\beta,\lambda-\nu}[\eta_2] = f_2(x)$$

where

$$\lambda = \frac{\mu+\nu}{2} - (\alpha-\beta) \ ,$$

$$f_1(x) = 2^{2\alpha} I_{\frac{\mu}{2}+\alpha,-\alpha}\{F_1(\xi)\xi^{-2\alpha};\ x\} \ ,$$

$$f_2(x) = 2^{2\beta} I_{\frac{\nu}{2}+\beta,-\beta}\{F_2(\xi)\xi^{-2\beta};\ x\}$$

and η_1 and η_2 are unknown functions which vanish on $(1,\infty)$. The operators appearing in (43) are the modified Erdelyi-Kober operators introduced by Sneddon [5] and are defined as follows: if $\alpha > 0$ then $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ denote the fractional integral operators

$$I_{\eta,\alpha}\{f(\xi);x\} = \frac{2}{\Gamma(\alpha)}\, x^{-2\eta-2\alpha} \int_0^x (x^2-\xi^2)^{\alpha-1}\xi^{2\eta+1}f(\xi)d\xi$$

$$K_{\eta,\alpha}\{f(\xi);x\} = \frac{2}{\Gamma(\alpha)}\, x^{2\eta} \int_x^\infty (\xi^2-x^2)^{\alpha-1}\xi^{-2\eta-2\alpha+1}f(\xi)d\xi;$$

whereas, if $\alpha < 0$, $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ are the fractional differential operators inverse to $I_{\eta+\alpha,-\alpha}$ and $K_{\eta+\alpha,-\alpha}$ respectively.

The system (43) may be regarded as a generalized Abel system. As was indicated in [3], only special cases of (43) fall within the class of Abel systems considered in that paper and for which simply closed form solutions are obtainable. In contrast, we show here that the full problem (43) may be treated by the methods of Section 3. Although this approach does not provide, in general, closed form solutions of (42), it does offer a means of obtaining useful theoretical information regarding the questions of existence and uniqueness. Moreover, the system (42) is ultimately reduced to a system of Fredholm equations of the second kind.

Two observations regarding the general character of the system (43) may be made immediately. The first is that both fractional integral and differential operators appear in (43). In particular, the four operators consist of two fractional integrals and their inverses. The second observation is that only when $\alpha = \beta$ does it occur that the unknown functions, η_i, appear in operators of the same order, i.e. in (43) the two integral operators have the same order and thus also their inverses. As will become apparent later, this greatly affects the tractibility of (43).

Without loss of generality we may assume $\nu > \mu$. Also, for simplicity we shall first assume that $\alpha = \beta$ and $\nu - \mu < 2$, which still includes all the physically interesting cases. Later, we shall indicate the necessary adjustments in the analysis to be made when these assumptions are relaxed.

Given these restrictions, equations (43) become

$$(44) \qquad \begin{aligned} &a_1 K_{\frac{\mu}{2}-\alpha,\frac{\nu-\mu}{2}}[\eta_1] + b_1 I_{\frac{\nu}{2},\frac{\mu-\nu}{2}}[\eta_2] = f_1(x) \\ &a_2 I_{\frac{\mu}{2},\frac{\nu-\mu}{2}}[\eta_1] + b_2 K_{\frac{\nu}{2}-\alpha,\frac{\mu-\nu}{2}}[\eta_2] = f_2(x) \end{aligned} \qquad 0 < x < 1$$

Introduction of

$$\phi_1(t) = t^{2\alpha-\mu+1}\eta_1(t) \ , \ \mu_1 = 1-(\nu-\mu)/2 \ , \ \alpha_1(t) = t^{\nu_1} \ ,$$

$$\nu_1 = (\mu+\nu)/2 - \alpha \ \text{ and } \ \beta_1(t) = 1$$

yields, in the notation of Section 3,

$$(45) \qquad K_{\frac{\mu}{2}-\alpha,\frac{\nu-\mu}{2}}[\eta_1] = \frac{2x^{\mu-2\alpha}}{\Gamma((\nu-\mu)/2)} K_1(\phi_1)$$

and

(46) $$I_{\frac{\mu}{2},\frac{\nu-\mu}{2}}[\eta_1] = \frac{2x^{-\nu}}{\Gamma(\frac{\nu-\mu}{2})} I_1(\phi_1) .$$

It should be noted that $\tilde{\alpha}_1$ and $\tilde{\beta}_1$, defined by (4), are now only power functions and that one of them is identically one. Moreover, the kernels $K_{1,1}$ and $K_{2,1}$ in (37) and (38) are easily seen to be given by

(47) $$\frac{\sin \mu_1\Pi}{\Pi} K_{1,1}(x,y) = \begin{cases} 0 & \nu_1 \leq 0 , \\ 2\nu_1(1-\mu_1)x^{2(\nu_1-1)}\,{}_2F_1(1-\nu_1,2-\mu_1;2;\frac{x^2-y^2}{x^2}) & \nu_1 > 0 , \\ & 0 < x < y \end{cases}$$

and

(48) $$\frac{\sin \mu_1\Pi}{\Pi} K_{2,1}(x,y) = \begin{cases} -2\nu_1(1-\mu_1)y^{-2(\nu_1+1)}\,{}_2F_1(1+\nu_1,\mu_1;2;\frac{y^2-x^2}{y^2}) & 0 < x < y , \\ & \nu_1 < 0 \\ 0 & \nu_1 \geq 0 , \; 0 < y < x . \end{cases}$$

Since solutions are sought for which the operators in (44) yield continuous functions,it follows that $t^{\nu}\eta_2(t)$ must vanish for $x = 0$ and $x = 1$. Hence we deduce that

(49) $$I_{\frac{\nu}{2},\frac{\mu-\nu}{2}}[\eta_2] = \frac{x^{-\mu}}{\Gamma(1-(\nu-\mu)/2)} \int_0^x (x^2-t^2)^{\frac{\mu-\nu}{2}} \frac{d}{dt}[t^{\nu}\eta_2(t)]dt$$

and

(50) $$K_{\frac{\nu}{2}-\alpha,\frac{\mu-\nu}{2}}[\eta_2] = \frac{-x^{\nu-2\alpha}}{\Gamma(1-(\nu-\mu)/2)} \int_x^1 (t^2-x^2)^{\frac{\mu-\nu}{2}} \frac{d}{dt}[t^{2\alpha-\mu}\eta_2(t)]dt .$$

Define $\nu_2 = \frac{\nu+\mu}{2} - \alpha$. If $\nu_2 = 0$, we define

$$\phi_2(t) = \frac{d}{dt}[t^\nu \eta_2(t)] , \alpha_2(t) = 1 , \beta_2(t) = 1 \text{ and } \mu_2 = \frac{\nu-\mu}{2}$$

and observe that

(51) $$I_{\frac{\nu}{2},\frac{\mu-\nu}{2}}[\eta_2] = \frac{x^{-\mu}}{\Gamma(1-\mu_2)} I_2(\phi_2)$$

and

(52) $$K_{\frac{\nu}{2}-\alpha,\frac{\mu-\nu}{2}}[\eta_2] = \frac{-x^{\nu-2\alpha}}{\Gamma(1-\mu_2)} K_2(\phi_2) ,$$

where $I_2(\phi_2)$ and $K_2(\phi_2)$ are given by (25).

If $\nu_2 \neq 0$, we define $\alpha_2(t) = t^{\nu_2}$ and

$$\phi_2(t) = t^{-2\nu_2} \frac{d}{dt}[t^\nu \eta_2(t)]$$

and note that

$$t^{2\alpha-\mu}\eta_2(t) = t^{-2\nu_2} t^\nu \eta_2(t) .$$

Line (51) is still valid but (52) must be amended. From the obvious identity

$$\frac{d}{dt}\,[t^{2\alpha-\mu}\eta_2(t)] = \phi_2(t) - 2\nu_2\, t^{-2\nu_2-1}[t^{\nu}\eta_2(t)]$$

we obtain

$$(53)\qquad K_{\frac{\nu}{2}-\alpha,\,\frac{\mu-\nu}{2}}[\eta_2] = \frac{-x^{\nu-2\alpha}}{\Gamma(1-\mu_2)}\; K_2(\phi_2)$$

$$- 2\nu_2 \int_x^1 \frac{t^{-2\nu_2-1}[t^{\nu}\eta_2(t)]}{(t^2-x^2)^{\mu_2}}\, dt \quad .$$

Moreover, it is straight forward to show that

$$(54)\qquad t^{\nu}\eta_2(t) = \frac{-\sin\eta_2\Pi}{\Pi}\left[\tilde{\alpha}_2(t^2)\int_t^1 \frac{2yk_2(y)dy}{(y^2-t^2)^{1-\mu_2}}\right.$$

$$\left. + \int_t^1 2yk_2(y)dy \int_t^y \frac{\tilde{\alpha}_2{}'(z^2)2z\,dz}{(y^2-x^2)^{1-\mu_2}}\right] .$$

We must now consider separately the two cases $\nu_2 > 0$ and $\nu_2 < 0$. For $\nu_2 > 0$ substitution of (54) into (53) yields

$$(55)\qquad K_{\frac{\nu}{2}-\alpha,\,\frac{\mu-\nu}{2}}[\eta_2] = \frac{-x^{\nu-2\alpha}}{\Gamma(1-\mu_2)}\; K_2(\phi_2)$$

$$- \int_x^1 \kappa_2(y)\, K_{3,2}(x,y)dy$$

where

(56) $$K_{3,2}(x,y) = \frac{2\nu_2}{\Gamma(1-\mu_2)}\, y^{2\mu_2-1}\, x^{\mu-2\alpha}$$

$$+ \frac{2 \sin \mu_2\Pi\, \nu_2^{\,2}}{\Pi\, \Gamma(2-\mu_2)}\, x^{\nu-2\alpha}(y^2-x^2)$$

$$\cdot \int_0^1 \frac{[x^2+\tau(y^2-x^2)]^{-2}}{\tau^{\mu_2-1}(1-\tau)^{1-\mu_2}}\, {}_2F_1(1+\nu_2,\ 1;\ 2-\mu_2;\ \frac{\tau(y^2-x^2)}{[x^2 1-\tau)+\tau y^2]})\, d\tau\ .$$

It should be noted that $x^{\nu}\, K_{3,2}(x,y) \,\varepsilon\, L^1(0,1)x(0,1)$ and is continuous for $0 < x \le y \le 1$.

If $\nu_2 < 0$ we obtain

(57) $$K_{\frac{\nu}{2}-\alpha,\frac{\mu-\nu}{2}}[n_2] = \frac{-x^{\nu-2\alpha}}{\Gamma(1-\mu_2)}\, K_2(\phi_2)$$

$$-\int_x^1 k_2(y) K_{4,2}(x,y)dy$$

with

(58) $$K_{4,2}(x,y) = \frac{2\,\nu_2\, x^{\nu-2\alpha}}{\Gamma(1-\mu_2)}\, y^{-\nu_2}\, {}_2F_1(1+\nu_2,\mu_2;1;\frac{y^2-x^2}{y^2})\ .$$

Moreover, we have $x^{\nu}K_{4,2}(x,y) \,\varepsilon\, L^1(0,1)x(0,1)$ and continuous for $0 < x \le y \le 1$. The expressions $I_2(\phi_2)$ and $K_2(\phi_2)$ appearing in (51), (52), (55), and (57) are given by (37) and (38) with

$$K_{1,2}(x,y) = \begin{cases} 0 & \nu_2 \leq 0, \quad 0 < x < y < 1 \\ \dfrac{2\nu_2(1-\mu_2)}{\sin \mu_2\pi}\, x^{2(\nu_2-1)} \; {}_2F_1(1-\nu_2, 2-\mu_2; 2; \dfrac{x^2-y^2}{x^2}) & \nu_2 > 0, \\ & 0 < y < x < 1 \end{cases}$$

and

$$K_{2,2}(x,y) = \begin{cases} 0 & \nu_2 \geq 0, \quad 0 < y < x < 1 \\ \dfrac{-2\nu_2(1-\mu_2)\,\pi}{\sin \mu_2\pi}\, y^{-2(\nu_2+1)} \; {}_2F_1(1+\nu_2, \mu_2; 2; \dfrac{y^2-x^2}{y^2}) & \nu_2 < 0 \\ & 0 < x < y < 1 . \end{cases}$$

The subsequent observations are valid for all values of ν_2. However, for definiteness we assume $\nu_2 > 0$. Substitution of (37) and (38) into (45), (46), (51) and (55) and from there into (44) yields a generalized Riemann system of the type

$$(59) \qquad A(x)\,\Phi^+(x) + B(x)\,\Phi^-(x) + K(\Phi^+,\Phi^-) = f(x) \qquad 0 < x < 1 .$$

Boundary equation (59) is then extended to all of $(-1,1)$ by the method of Section 3. Alternatively, (59) may be transformed into the system of singular integral equations

$$(60) \qquad S(x)\,\psi(x) + T(x)\,\frac{1}{\pi i} \int_{-1}^{1} \frac{\psi(t)dt}{(t-x)} + \hat{R}(\psi) = f(x) \qquad -1 < x < 1$$

by introducing

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{\psi(t)dt}{(t-z)} .$$

In (60),

$$S(x) = \frac{A(x) - B(x)}{2}$$

and

$$T(x) = \frac{A(x) + B(x)}{2} ,$$

and $\hat{K}(\psi)$ is a Fredholm operator. Examination of the dominant singular part of (60), or equivalently the principal part of (59), yields important theoretical information about (44). For example, the number of solutions of (59) (and hence of (44)) is at least as large as the number of solutions of the dominant homogeneous singular equation

$$(61) \qquad S(x)\,\psi(x) + T(x)\,\frac{1}{\pi i}\int_{-1}^{1}\frac{\psi(t)dt}{(t-x)} = 0 ,$$

or its corresponding Riemann problem

$$(62) \qquad A(X)\,\Phi^{+}(x) + B(x)\,\Phi^{-}(x) = 0 .$$

Examination of (61) or (62) will thus provide conditions necessary for the existence of a unique solution to the dual relations (42). It therefore becomes necessary to compute the index of the system (62), which from the general theory of Muskhelishvili [4], is most easily determined by actually solving (62). This can be accomplished by transforming (62) in the usual way [4] to a system of Fredholm equations of the second kind and then iteratively constructing the solutions. However, in certain cases the technique presented in [3] for uncoupling systems of the type (60) into two ordinary uncoupled Riemann problems will provide simple closed form solutions. To decide the applicability of the method of [3] to (62) we must examine more closely the matrices $A(x)$ and $B(x)$, the components of which are given by $A(x) = (a_{ij}(x))$, $B(x) = \overline{A(x)}$, $A(-x) = -\overline{A(x)}$ and on $0 < x < 1$

$$a_{11}(x) = -a_1\, x^{2\nu_2-\nu}[1-x^2]^{1/2-\mu_1}\, \frac{\Gamma(\mu_1)}{\Pi}\, e^{-\mu_1 \Pi i} ,$$

$$a_{12}(x) = b_1\, x^{2\nu_2-\mu}[1-x^2]^{1/2-\mu_2}\, \frac{\Gamma(\mu_2)}{2\Pi} ,$$

$$a_{21}(x) = a_2\, x^{2\nu_2-\mu}[1-x^2]^{1/2-\mu_1}\, \frac{\Gamma(\mu_1)}{\Pi} ,$$

$$a_{22}(x) = b_2\, x^{2\nu_2-\mu}[1-x^2]^{1/2-\mu_2}\, \frac{\Gamma(\mu_2)}{2\Pi}\, e^{-\mu_2 \Pi i} .$$

The first restriction to be placed on $A(x)$ is that $\det(A)(x) \neq 0$ (except perhaps for $x = 0$, ± 1). Hence it is assumed that

$$(63) \qquad a_1\, b_2\, e^{-\Pi i(\mu_1+\mu_2)} + a_2 b_1 \neq 0 .$$

Recalling that $\mu_1 + \mu_2 = 1$, we see that (63) is equivalent to

$$(64) \qquad \delta \equiv a_1\, b_2 - a_2\, b_1 \neq 0 .$$

It follows that when (64) holds, the system (62) is equivalent to

$$(65) \qquad \Phi^+(x) = -\frac{1}{\delta}\, G(x)\, \Phi^-(x) ,$$

where $G(x) = A^{-1}(x)\,\overline{A(X)} = (g_{ij}(x))$, $G(-x) = \overline{G(x)}$ and on $(0,1)$

$$g_{11}(x) = a_2 b_1 + a_1 b_2\, e^{\Pi i(\mu_1 - \mu_2)} ,$$

$$g_{12}(x) = b_1\, b_2\, x^{\nu-\mu}[1-x^2]^{\mu_1-\mu_2}\, \frac{\Gamma(\mu_2)}{\Gamma(\mu_1)}\, i \sin \mu_2\Pi ,$$

$$g_{21}(x) = -a_1\, a_2\, x^{\mu-\nu}[1-x^2]^{\mu_2-\mu_1}\, \frac{\Gamma(\mu_1)}{\Gamma(\mu_2)}\, 4\, i \sin \mu_1\Pi ,$$

$$g_{22}(x) = \overline{g_{11}(x)} .$$

Uncoupling (65) requires finding a matrix $P(z)$ analytic in the plane cut along $(-1,1)$ and such that

$$P(x)\, G(x)\, P^{-1}(x) = D(x) \qquad -1 < x < 1$$

where $D(x)$ is diagonal. As was shown in [3] , $P(x)$ must be of the form

$$P(x) = c(x) \begin{pmatrix} g_{12}(x) & -\sqrt{g_{12}(x)\, g_{21}(x)} \\ g_{12}(x) & \sqrt{g_{12}(x) g_{21}(x)} \end{pmatrix}$$

where $c(x)$ is a scalar function. Since

$$\sqrt{g_{12}(x) g_{21}(x)} = 2\sqrt{\sin \mu_1\Pi \sin \mu_2\Pi\, a_1 a_2 b_1 b_2} ,$$

the matrix $P(x)$ has an analytic extension to the cut plane if and only if $x^{\mu-\nu}[1-x^2]^{\mu_2-\mu_1}$ has no branch point at infinity. Recalling that $\mu_1 - \mu_2 = 1 - (\nu-\mu)$ and $0 < \nu-\mu < 2$ we conclude that (65) can be uncoupled if and only if $\nu-\mu = 1$. This condition is satisfied by the systems arising in applications to bimedia crack problems in elasticity [1]. When the restriction $0 < \nu-\mu < 2$ is withdrawn, it becomes apparent that (65) uncouples whenever $\nu-\mu$ is an odd integer. When $\nu-\mu$ is an even integer, the original system (43) was shown in [3] to reduce to a single linear ordinary differential equation. Hence, it is apparent that (43), is much more easily analysed when $\nu-\mu$ is an integer than it is otherwise. Moreover, a simple closed form solution is obtained whenever $\nu_1 = \nu_2 = 0$. It now is a simple matter to compute the indices of the two uncoupled Riemann problems and obtain conditions for the solvability of (44). In particular, the indices provide necessary conditions for uniqueness to hold for the dual relations (42).

It remains to consider how removing the restrictions $0 < \nu-\mu < 2$ and $\alpha = \beta$ affects the analysis of (44). Maintaining $\alpha = \beta$ but allowing $\nu-\mu$ to be any positive number does not affect the general character of (44) and requires only minor alterations in the analysis presented above. Since the calculations involved are rather tedius we shall dispense with a detailed analysis of this case. However, if $\alpha \neq \beta$ the behavior of (43) is substantially different from that of (44). To illustrate the difference we shall consider (43) with $\alpha > \beta$, and for simplicity we assume $0 < \nu-\mu < 2$. Note that in this case $\nu - \lambda > 0$, and we may define n to be the least positive integer greater than $\nu - \lambda$, i.e. $n-1 < \nu-\lambda \leq n$. To avoid a case argument we shall assume $n-1 < \nu-\lambda < n$ and $n \geq 2$.

The following identity is easily verified

$$K_{\frac{\mu}{2}-\alpha,\nu-\lambda}(\eta_1) = \frac{4x^{\mu-2\alpha}}{\Gamma(\frac{\nu-\mu}{2})\Gamma(\alpha-\beta)} \int_x^1 (t^2-x^2)^{\alpha-\beta-1} t\,dt$$

$$\cdot \int_t^1 (y^2-t^2)^{\frac{\nu-\mu}{2}-1} y^{2\beta-\nu+1}\,\eta_1(y)dy \; .$$

Therefore, if we define $\nu_1 = \frac{\mu+\nu}{2} - \beta$, $\alpha_1(t) = t^{\nu_1}$, $\beta_1(t) = 1$, $\mu_1 = 1 - \frac{(\nu-\mu)}{2}$ and $\phi_1(t) = \eta_1(t)\, t^{2\beta-\nu+1}$ we obtain

(66) $$I_{\frac{\mu}{2},\frac{\nu-\mu}{2}}(\eta_1) = \frac{2x^{-\nu}}{\Gamma(1-\mu_1)} I_1(\phi_1)$$

and

(67) $$K_{\frac{\mu}{2}-\alpha,\nu-\lambda}(\eta_1) = \frac{4\,x^{\mu-2\alpha}}{\Gamma(1-\mu_1)\Gamma(\alpha-\beta)} \int_x^1 (t^2-x^2)^{\alpha-\beta-1}\, t\, K_1(\phi_1)dt$$

where $I_1(\phi_1)$ and $K_1(\phi_1)$ are defined as in (25).

Moreover, it is straight forward to show that

(68) $$K_{\frac{\mu}{2}-\beta,\lambda-\nu}(\eta_2) = \frac{2(-1)^n}{\Gamma(\lambda-\nu+n)} x^{\nu-2\beta} \int_x^1 (t^2-x^2)^{\lambda-\nu+n-1} t D_t^n[t^{2\alpha-\mu}\eta_2(t)]dt$$

and

(69) $$I_{\frac{\nu}{2},\frac{\mu-\nu}{2}}(\eta_2) = \frac{2x^{-\mu-2}}{\Gamma(\frac{\mu-\nu}{2}+1)} \int_0^x (x^2-t^2)^{\frac{\mu-\nu}{2}} t^{\nu-\mu+1} D_t[t^{\mu+2}\beta_2(t)]dt \; ,$$

where $D_t = \left(\frac{1}{2t}\frac{d}{dt}\right)$. Now let $\phi_2(t) = t\, D_t^n[t^{2\alpha-\mu}\eta_2(t)]$, $\alpha_2(t) = 1$

$\beta_2(t) = 1$ and $\mu_2 = 1 - n + \nu - \lambda$, and let $K_2(\phi_2)$ and $I_2(\phi_2)$

be as in (25). We then have from (68)

(70) $$K_{\frac{\nu}{2} - \beta, \lambda-\nu}(\eta_2) = \frac{2(-1)^n x^{\nu-2\beta}}{\Gamma(1-\mu_2)} K_2(\phi_2)$$

and after some manipulation (69) becomes

(71) $$I_{\frac{\nu}{2}, \frac{\mu-\nu}{2}}(\eta_2) = \frac{(\mu+1-\alpha)\; 8x^{-\mu-2}}{\Gamma(1-\mu_2)\Gamma(\mu_1)\Gamma(\nu-\lambda)} \int_0^x y I_2(\phi_2)dy \int_y^x (x^2-t)^{\frac{\mu-\nu}{2}} (t^2-y^2)^{\nu-\lambda-1}$$

$$.t^{\nu+\mu-2\alpha+1}dt$$

$$+ \frac{8x^{-\mu-2}}{\Gamma(1-\mu_2)\Gamma(\mu_1)\Gamma(\nu-\lambda-1)} \int_0^x y I_2(\phi_2)dy \int_y^x (x^2-t^2)^{\frac{\mu-\nu}{2}} (t^2-y^2)^{\nu-\lambda-2}$$

$$.t^{\nu+\mu-2\alpha+3}dt$$

$$= \int_0^x I_2(\phi_2) K_{2,1}(x,y)dy \quad ,$$

where

$$K_{2,1}(x,y) = \frac{(4yx^{\nu-2\alpha-4}(x^2-y^2)^{\alpha-\beta-1})}{\Gamma(1-\mu_2)\Gamma(\alpha-\beta)}$$

$$\cdot \left\{ \frac{(\mu+1-\alpha)}{(\alpha-\beta)} x^2(x^2-y^2)\; {}_2F_1(\alpha - \frac{(\mu+\nu)}{2}, 1 - \frac{(\nu-\mu)}{2}; (\alpha-\beta)+1; \frac{(x^2-y^2)}{x^2}\right.$$

$$\left. + {}_2F_1(\alpha - \frac{(\mu+\nu)}{2} - 1, 1 - \frac{(\nu-\mu)}{2}; (\alpha-\beta); \frac{(x^2-y^2)}{x^2}) \right\} .$$

Once again a generalized Riemann boundary value problem equivalent to (42) is obtained by substituting (32), (33), (37), and (38) into (43). However, in this case the resulting system has a singular (or degenerate) principal part. In particular, the system has the form

$$A(x)\ \Phi^{+}(x) + B(x)\ \Phi^{-}(x) + k(\Phi^{+},\Phi^{-}) = F(x)$$

with $A(x) = (a_{ij}(x))$, $B(x) = \overline{A(x)}$, $a_{11}(x) = a_{12}(x) = 0$, $a_{21}(x) = a_2 x^{-\nu}\tilde{\alpha}_1(x^2)(1-x^2)^{1/2-\mu_1}/(\Gamma(1-\mu_1)\sin\mu_1\Pi)$ and $a_{22}(x) = b_2(-1)^{n+1}x^{\nu-2\beta}e^{-\mu_2\Pi i}(1-x^2)^{1/2-\nu_2}/(\Gamma(1-\mu_2)\sin\mu_2\Pi)$. It is now apparent that $A(x)$ and $B(x)$ are not invertible and the methods of this section do not effect a simplification of (43), or hence of (42). Evidently, the behavior of (42) is substantially different for $\alpha = \beta$ and $\alpha \neq \beta$. This is due to the fact that when $\alpha \neq \beta$, the function η_1 in (43) appears in operators of different order, as does η_2 . For $\alpha = \beta$ this does not occur. That this is important is easily illustrated by considering a single generalized Abel equation of the type

$$(72)\qquad a(x)\int_0^x \frac{\phi(t)dt}{(x-t)^{\mu_1}} + b(x)\int_x^1 \frac{\phi(t)dt}{(t-x)^{\mu_2}} = f(x) \qquad 0 < x < 1 .$$

Applying the methods of this paper to (72) shows that (72) is equivalent to the singular integral equation

$$A(x)\ \Psi(x) + B(x)\ \frac{1}{i\Pi}\int_0^1 \frac{\psi(t)dt}{(t-x)} + k(\psi) = F(x) .$$

If $\mu_1 = \mu_2 = \mu$ then

$$A(x) = i \tan \frac{\pi}{2}\mu \, (a(x)+b(x)) ,$$

$$B(x) = (a(x) - b(x))$$

and

$$k(\psi) \equiv 0 .$$

However, if $\mu_1 \neq \mu_2$, say $\mu_1 > \mu_2$, then

$$A(x) = i \tan \left(\frac{\pi}{2} \mu_1 \right) a(x) ,$$

$$B(x) = a(x)$$

and

$$k(\cdot) \not\equiv 0 ,$$

The integrals in (72) correspond to fractional integral operators of order $1 - \mu_1$ and $1 - \mu_2$ respectively. Thus, if the orders of the operators in (72) are the same the essential structure of the equation is governed by the functions $a(x) + b(x)$ and $a(x) - b(x)$; whereas when the orders are different, the fundamental properties of (72) are determined only by the coefficient of the operator of lowest order. This is analogous to the characteristic behavior of a differential equation being determined by the coefficients of the terms of highest order.

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A method is presented for solving certain systems of generalized Abel integral equations by constructing equivalent singular integral equations and their corresponding Riemann boundary value problems. An application is then given to a class of simultaneous dual relations of a type arising in bimedia fracture problems in elasticity and viscoelasticity. The equations discussed in this paper are a generalization of those considered in an earlier paper of Lowengrub and Walton [3].

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